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A Family of Location Models for Multiple-Type Discrete Dispersion

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One of the defining objectives in location science is to maximize dispersion. Facilities can be dispersed for a wide variety of purposes, including attempts to optimize competitive market advantage, disperse negative impacts, and optimize security. With one exception, all of the extant dispersion models consider only one type of facility, and ignore problems where multiple types of facilities must be located. We provide examples where multiple-type dispersion is appropriate and based on this develop a general class of facility location problems that optimize multiple-type dispersion. This family of models expands on the previously formulated definitions of dispersion for single types of facilities, by allowing the interactions among different types of facilities to determine the extent to which they will be spatially dispersed. We provide a set of integer-linear programming formulations for the principal models of this class and suggest a methodology for intelligent constraint elimination. We also present results of solving a range of multiple-type dispersion problems optimally and demonstrate that only the smallest versions of such problems can be solved in a reasonable amount of computer time using general-purpose optimization software. We conclude that the family of multiple-type dispersion models provides a more comprehensive, flexible, and realistic framework for locating facilities where weighted distances should be maximized, when compared with the special case of locating only a single type of facility.

Introduction

One of the defining objectives in location science is to maximize dispersion. Facilities can be dispersed for a wide variety of purposes, including keeping competitors of the same franchise system apart, dispersing criminal rehabilitation facilities from population centers, and locating nuclear power plants in such a way as to maximize security. Even though a range of general dispersion models have been developed, only one involves a combination of different types of facil-

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ities. This model is the regional energy facility location model of Church and Cohon (1976). In that model, two different types of facilities, fossil fuel power plants and nuclear power plants, were located simultaneously in a region where a number of objectives were optimized, including dispersive terms. In the general case, when locating facilities with an objective of maximizing dispersion, there may be more than one type of facility to locate. For example, a dispersed configuration of missile launch facilities should not coincide with the location of a dispersed pattern of radio towers. A strike against one facility would compromise the others. The motivation for this work is based upon the fact that locating a facility of one type may well be determined or influenced by the location of facilities of other types. The primary objective of this article is to define a family of models for the simultaneous location of multiple types of facilities where the facilities differ from one another in some fundamental way, and these differences have an influence on the measure of optimal dispersion.

In order to form the basis for this new family of models, we review the recent location science literature involving the location of dispersed facilities. A set of applications for dispersion models is described, several ways in which dispersion can be measured are outlined, and several special cases of dispersion models are recognized. Most importantly, this review highlights the long-recognized need for (and lack of) discrete dispersion models involving multiple-type facilities. In the third section this need is addressed through the presentation of a family of multiple-type discrete dispersion models that builds on the existing foundation of discrete dispersion in location science. A set of general models is presented, and special cases are identified. In the fourth section we present some computational experience in solving this new class of models using a data set originally presented in Kuby (1987). A summary of our results and a discussion of areas for future research are provided in the sixth section.

A review of dispersion models in location science

Applications for dispersion models

Dispersion models can be applied over a spectrum of scales: macroscale applications include such things as the location of radio transmitters or defense installations over a large geographic region; mesoscale applications include the location of schools, housing developments, landfills, or incinerators within a smaller, well-defined geographic region; and microscale applications of dispersion can include such things as product shelf location and factory or classroom layout studies.

By far the most common use of dispersion models is for the location of undesirable facilities (Church and Garfinkel 1978; Drezner and Wesolowsky 1985; Erkut and Neuman 1989; Drezner and Wesolowsky 1996). This literature is further divided into the location of noxious and obnoxious facilities. Noxious facilities are those that present some health risk to any population that would be exposed to either the damaging repercussions of an accident at the facility or the damaging consequences of long-term exposure to the facility. Examples of noxious facilities include coal-fired power stations, nuclear power plants, hazardous waste storage sites, oil storage tanks, ammunition dumps, landfills, and incinerators. Obnoxious facilities are not expected to cause health risks to populations, but they may have (or be perceived to have) deleterious social or economic consequences associated with their location and operation. Examples of obnoxious facilities include prisons, activities that generate excessive noise, social service centers, and rehabilitation (e.g., drug treatment) centers (Murray et al. 1998). Obnoxiousness may result in disagreements between the facility operator and the local population that are based on ideological or attitudinal conflict (Sorensen, Soderstrom, and Carnes 1984). Facilities that are considered undesirable may have attributes that are both noxious and obnoxious.

Metrics of dispersion

If dispersion is a reasonable objective to optimize, one may choose from several known ways to quantify optimal dispersion. Erkut and Neuman (1990) made a comparison of dispersion models based on the way these models defined maximal dispersion. They employed a three-syllable naming convention to distinguish between different types of dispersion. Using this convention, the first syllable for each model was ''Max'' denoting that all models attempt to maximize the amount of dispersion among selected facility sites. Both the second and third syllables were either ''sum'' or ''min.'' In the second syllable, a ''sum'' operator indicates a concern with overall system performance while a ''min'' operator indicates a concern for worst-case performance. In the third syllable ''sum'' or ''min'' refers to the facility interactions considered for each facility. When the ''min'' operator is used, the objective function is constrained by the minimum distance between each facility and any of its neighbors in a given solution. The ''sum'' operator indicates that the distances from each facility to all other facilities will constrain the objective function. In other words, the nomenclature tells one to maximize (1st operator) . . . the sum or minimum (3rd operator) . . . of the summed or minimized distances (2nd operator) between facilities.

Using this system, there are four possible permutations. The first of these is the MaxMinMin problem, which seeks to maximize the minimum distance between any two located facilities. This problem is referred to in the literature as the *p*-dispersion problem (Moon and Chaudhry 1984) and was first developed as an extension to the *p*-center problem (Shier 1977). Shier demonstrated that the $p+1$-dispersion problem is the dual of the *p*-center problem. An optimal solution to the *p*-dispersion problem will locate *p* facilities such that the minimum distance between any pair of facilities is maximized. Dispersion as defined by the MaxMinMin objective is concerned with any pair of locations that might be located near each other and it will avoid such a situation at the expense of the overall system spread. The NP-completeness of this problem has been demonstrated by a reduction to the clique problem (Erkut 1990). Additional formulations of this model with examples of applications and solution procedures are available in the literature (Chandrasekaran and Daughety 1981; Kuby 1987).

A second definition of maximal dispersion is given by the MaxSumMin problem. This problem seeks to find a maximally dispersed set through the use of the sum of the minimum distances between located facilities. This problem is known in the literature as the *p*-defense problem and was identified by Moon and Chaudhry (1984). With this dispersion objective two or more facilities could be optimally located near to one another if doing so allows other facilities to be located extremely far away from their nearest neighbors, causing the sum of the minimum distances to be larger. The measurement of performance is, however, still based on the minimum distance between a facility and any of its neighbors.

The MaxMinSum objective deviates from the previous two objectives in that it does not consider only the minimum distance between a located facility and its nearest neighbor, but instead measures the distance between that facility and all other located facilities. This distance can be thought of as a hub distance where the facility located at site *i* is at the hub of a wheel, and the spokes of the wheel radiate out from *i* to all other located facilities (Fig. 1). The MaxMinSum problem was unknown in the literature until the review of dispersion objectives by Erkut and Neuman (1990).

The fourth objective seeks the most global measure of dispersion yet presented. The MaxSumSum problem shares the concept of the hub distance with the MaxMinSum problem, but expands on that concept to maximize the sum of all of the

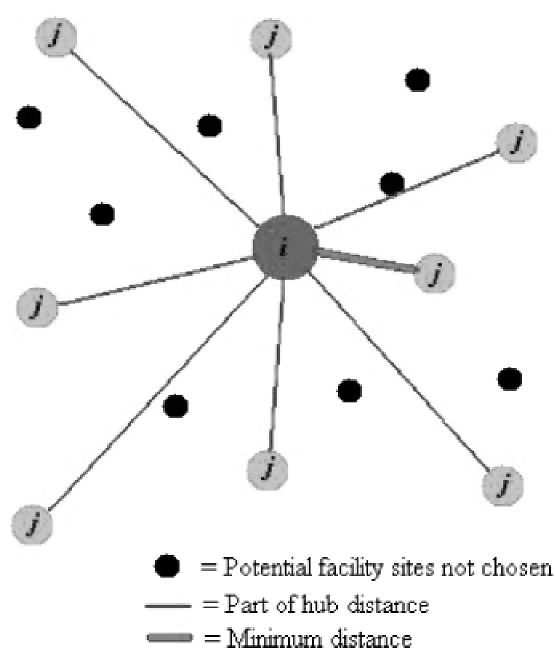

Figure 1. Hub distance versus minimum distance.

hub distances for every located facility, rather than just trying to maximize the smallest of those hub distances. This problem was first identified as an extension to the one-facility maximum median (maxian) problem on a network (Church and Garfinkel 1978). Termed the *p*-maxian problem, this model is described as an effort to locate *p* facilities simultaneously far from a given set of nodes and also far from each other. This definition hints at the need for multiple-type dispersion models in that fixed locations, demand centers, and new facilities must be considered. Nine years later a formulation was provided for the discrete maxisum problem, which was the first to cast the problem in the context of dispersion (Kuby 1987). Kuby's formulation seeks to locate *p* facilities among *n* discrete nodes so as to maximize the sum of distances between located facilities. Subsequently, a quadratic integer-programming and a linear integer-programming formulation have been presented, as well as both exact and heuristic solution procedures for special cases of this problem (Erkut, Baptie, and von Hohenbalken 1990).

In summary, the basic dispersion models presented by Erkut and Neuman are based on different objectives, and can therefore result in dramatically different location patterns, all of which can be considered dispersed. These models are all basic in the sense that there are no minimum or maximum required distances between facilities, there are no constraints on the number of facilities to be located (the value of *p*) except the number of available facility locations, and there is no distinction between new and existing facilities. It should be noted that, despite the clear differences in the objectives and potential outcomes of the four objectives, it is possible that, for particular instances of the problem, the difference between the solutions generated by the different models may not be significant (Erkut and Neuman 1990).

Special case models of dispersion

A number of dispersion models have been presented in the location science literature to address more specific instances of dispersion. These models consider additional constraints on—or extensions to—some parameter of the more basic models presented above. One of these, the *r*-separation problem, considers the case, where as many facilities as possible must be located at least a given minimum distance away from each other. That is, the minimum acceptable level of dispersion is known in advance. Also known as the anticover problem (Moon and Chaudhry 1984), the *r*-separation problem is known to be NP-complete due to its equivalence to the independent set problem in graph theory. A recent review of the *r*-separation problem (Erkut and ReVelle 1996) provides six formulations and tests them for computational efficiency.

As discussed in the context of the MaxSumSum problem above, another group of special case discrete dispersion models include both new and existing facilities in the determination of optimal location. This problem has been termed the *p*-anticenter-dispersion model and formulated as a MaxMinMin problem (Erkut 1990). The *p*-anticenter-dispersion problem can be reduced to the *p*-dispersion problem and thus is NP-complete as well. Although existing facilities can be considered a second type of facility in comparison with a set of new facilities, the *p*-anticenter-dispersion problem still only locates a single type of facility. The locations of existing facilities are considered when making the decisions to site new facilities, but the model has no impact on their placement, and there is no differentiation of facility type between new and existing facilities. That is, the only difference between new and existing facilities in the *p*-anticenter-dispersion problem is the time at which they are located.

There also exist several models that involve different forms of dispersion metrics, other than the four basic models already described. One of these special cases includes a formulation and solution procedure for locating *a single point* such that the minimum distance from a given set of points is maximized (Dasarathy and White 1980). Drezner and Wesolowsky (1985) consider a problem related to the *r*-separation problem, where the dispersion of facilities is desirable but a given *maximum* distance must not be exceeded . Other examples include a consideration of facility interaction (Welch and Salhi 1997), impact models (Murray et al. 1998), and risk-sharing models for locating undesirable facilities (Ratick and White 1988).

All of the basic dispersion models discussed by Erkut and Neuman (1990) consider the location of *p* facilities among *n* possible facility sites. All of the facilities to be located are of the same type, and there are no fixed facilities or other sites representing interaction with the facilities (e.g., demand, population, or conflicting uses). The formulations for the 1-Maxian on a network (Church and Garfinkel 1978) and the discrete *p*-Maxian problem with existing facilities (Erkut, Baptie, and von Hohenbalken 1990) demonstrate that the most general dispersion model must incorporate variance in both the number of facilities to be located and the number of *types of facilities* to be located or considered in the solution. In other words, this review of dispersion models in location science demonstrates that, although there is a diverse set of models that cover a range of dispersion objectives, there is a need for models that can locate two or more different types of undesirable facilities simultaneously. The following section addresses this need by formulating a family of multiple-type dispersion models.

The family of multiple-type dispersion models

In order to formulate a family of multiple-type dispersion models, we begin by describing a method for capturing the differences between types of facilities. This concept is used to derive formulations for multiple-type dispersion models with several objectives. Additional special-case formulations are provided and as multiple-type dispersion models contain extensive constraint sets, a methodology for the intelligent elimination of specific constraints will be explored.

Measure of repulsion

Intrinsic to the concept of multiple-type dispersion is the idea that facilities of different types vary in the extent to which they ought to be dispersed. We term this difference "repulsion." If facilities of different types are identical in their repulsion from one another, then they can be considered to be of the same type. Although it is somewhat unintuitive, a smaller repulsion measure reflects a relatively greater significance in the facility interactions than a larger repulsion measure does. Recall that we are maximizing repulsion-weighted distance. As a smaller repulsion measure creates smaller repulsion weighted distances, this value is more binding on the objective function value and reflects greater importance in terms of facility interaction. An example to demonstrate how repulsion measures can influence location decisions is illustrated in Fig. 2.

In this example there are four potential locations for facilities at the corners of the unit square. Two incinerators and two parks must be allocated to these discrete locations in such a way as to maximize the minimum repulsion-weighted distance between any two facilities (the MaxMinMin objective). All of these facility types should be dispersed: parks should be dispersed to promote access to such amen-

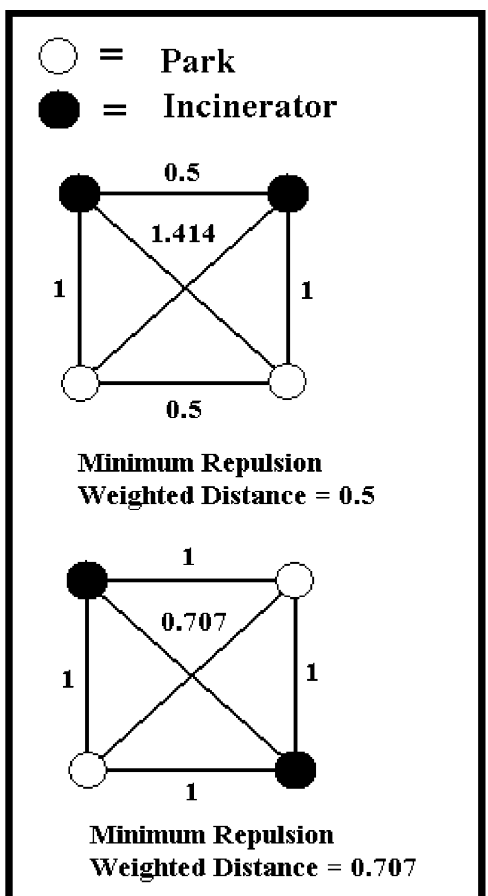

Figure 2. Solutions based on repulsion-weighted distance.

ities, incinerators should be dispersed to avoid excessive air pollution in any one area, and parks should be dispersed from incinerators to avoid health hazards to park patrons. Without any repulsion measures, all possible solutions give the identical objective function value. Assume, though, that the repulsion measure (i.e., repulsion weight or metric per unit distance) between two incinerators or two parks is 0.5, and the repulsion measure between an incinerator and a park is 1. These values suggest that it is more important to disperse incinerators and parks among themselves than it is to disperse incinerators vis-à-vis parks. There are only two significant solutions to this problem: (1) the facilities of like type are located adjacent to each other or (2) they are located diagonally across the unit square. In the first case there are two instances where the park-to-incinerator distance is 1, two cases where the park-to-incinerator distance is 1.414, and both the park-to-park distance and the incinerator-to-incinerator distance are 1. If these distances are multiplied by the appropriate repulsion weights, the smallest value is generated by the park–park distance of 1.00 and the incinerator–incinerator distance of 1.00, each multiplied by a repulsion weight of 0.5 yielding a value of 0.5. Thus, the minimum repulsion-weighted distance between any two facilities is 0.5. In the second case all four park-to-incinerator distances are 1 and both the park-to-park and incinerator-to-incinerator distances are 1.414. The minimum repulsion-weighted distance is then 0.5×1.414 or 0.707. As this minimum value of 0.707 is greater than the minimum value of 0.5 in the first case, the second solution is the optimal solution for this problem instance.

In this example it is solely the measure of repulsion among the different types of facilities that influences the optimal location pattern. Generally speaking, the notion of a repulsion measure is simply a way of differentiating between types of facilities, and quantifying the interaction between those types. Each pairing of facility types is assigned a value that reflects the extent to which the objective function suffers as the distance between these two types of facilities decreases.

Naming convention and notation

With a structure in place to differentiate between types of facilities, we can formulate a series of multiple-type dispersion models. The three principal variables (number of facilities (p), number of types of facilities (t), and number of potential facility sites (n)) are used to construct a naming convention for dispersion problems. As an example, the p–t–n dispersion problem seeks to locate p facilities of t types among n potential facility sites.

The formulations of these problems employ the following notation:

t = number of types of facilities
n = number of potential facility sites
K, L = indices for facility types
i, j = indices for potential facility sites
p_K = number of facilities to locate of type K
Z = objective function to maximize
x_i^K = 1 if a facility of type K is located at candidate site i, 0 otherwise

d_{ij} = the distance between candidate sites i and j
Q^{KL} = measure of repulsion between facilities of type K and of type L per unit distance of separation
M = number larger than the largest distance between any two sites
$P = \sum_{K=1}^{t} p_K$

The *p–t–n* dispersion problem (general multiple-type dispersion problem)

The first formulation of the p–t–n dispersion problem seeks to locate p_K facilities of each of t different types, such that the minimum repulsion-weighted distance between any two facilities of any type is maximized. This corresponds to the single-type MaxMinMin formulation identified by Erkut and Neuman (1990), which is an extension of the formulation of Kuby (1987).

$$\text{Max } Z \tag{1}$$

Subject to:

$$Z \leq Q^{KL} d_{ij} + M(2 - x_i^K - x_j^L), \quad i \text{ and } j = 1, 2, \ldots, n, \quad i \neq j, \quad K \text{ and } L = 1, 2, \ldots, t, \quad L \geq K \tag{2}$$

$$\sum_{K=1}^{t} x_i^K \leq 1, \quad i = 1, 2, \ldots, n \tag{3}$$

$$\sum_{i=1}^{n} x_i^K = p_K, \quad K = 1, 2, \ \ldots, \ t \tag{4}$$

$$x_i^K = 0 \ \text{ or } \ 1, \quad i = 1, 2, \ldots, n, \quad K = 1, 2, \ldots, t \tag{5}$$

Constraints (2) force the value of the objective function Z to be less than or equal to the minimum of the repulsion-weighted distances between any two facilities of any type. A constraint exists for each pairing of potential facility locations and each pairing of two types of facilities, with the exception that two facilities of the same type cannot logically be located at the same location; that is, when both i and j refer to the same potential facility location, the placement of two facilities of the same type at that location would result in a weighted distance of 0, creating an upper bound of 0 on the objective function. Additionally, it is not necessary to include constraints when the value of L is less than the value of K, as doing so would result in duplicate constraints. If either (or both) of the two facility locations for a given constraint do not contain a facility of the type under consideration (if x_i^K or x_j^L are equal to zero), then the objective function value Z need only be less than or equal to a very large number added to the repulsion-weighted distance between the facilities. When both potential facility sites under consideration are assigned a facility of the types under consideration, the term containing the very large number, M, is equal to 0, and Z is constrained only by the repulsion-weighted distance between the facilities. As a constraint exists for all logical pairings of potential facility locations, Z must be less than or equal to the minimum weighted distance between any two facilities of any type. The sense of maximization in the objective function (1) ensures that a solution will be sought which maximizes this minimum weighted distance.

Constraints (3) ensure that only one type of facility can be located at any particular facility site. Constraint (4) ensures that exactly p_K facilities of type K will be located, and constraints (5) require that all decision variables are equal to either 0 or 1, guaranteeing an integer solution.

The MaxSumMin formulation of the p–t–n dispersion problem seeks to maximize the sum of the minimum repulsion weighted distances associated with each of the selected facility locations and types and can be formulated as follows:

$$\text{Max} \sum_{i=1}^{n} Z_i \tag{6}$$

subject to:

$$Z_i \leq Q^{KL} d_{ij} + M(1 - x_j^L) \quad i \text{ and } j = 1, 2, \ldots, n, \quad i \neq j, \ K \text{ and } L = 1, 2, \ldots, t \tag{7}$$

$$Z_i \leq M \sum_{K=1}^{t} x_i^K, \quad i = 1, 2, \ldots, n \tag{8}$$

$$\sum_{K=1}^{t} x_i^K \leq 1, \quad i = 1, 2, \ldots, n \tag{9}$$

$$\sum_{i=1}^{n} x_i^K = p_K, \quad K = 1, 2, \ldots, t \tag{10}$$

$$x_i^K = 0 \text{ or } 1, \quad i = 1, 2, \ldots, n, \quad K = 1, 2, \ldots, t \tag{11}$$

This formulation differs from the MaxMinMin formulation in that a value Z_i — representing the minimum repulsion-weighted distance from each i to another facility location—is determined, and the maximum sum of these Z_i values is the objective. Constraints (7) differ from constraints (2) in the MaxMinMin formulation in several respects. First the variable Z_i is determined for each potential facility site i rather than a global Z value that pertains to all facility sites. As in constraints (2), constraints (7) force the value of Z_i to be less than or equal to the minimum of the repulsion-weighted distances between a facility of some type located at i and any other facility of any type. When a facility of type L is located at potential facility site j, Z_i is constrained by the minimum repulsion-weighted distance from i to j. When there is no facility of type L at potential facility site j ($x_j^L = 0$), Z_i is constrained only by the repulsion-weighted distance between i and j, in addition to a very large number, M. In order to ensure that a significant Z_i value is generated only when a facility is located at i, constraints (8) force Z_i to be equal to zero if there is no facility of any type located at i. As the objective function seeks to maximize the sum of Z_i values, these cases will not affect the optimal solution value. Note that in contrast to the MaxMinMin formulation, a constraint of type (7) must be generated for all combination of types K and L in order that the repulsion measures Q^{KL} can be applied to the appropriate combinations of facility types. As an example, although Q^{12} and Q^{21} are still equal, the value of $Q^{12}d_{ij} + M(1 - x_j^2)$ is not necessarily equal to the value of $Q^{21}d_{ij} + M(1 - x_j^1)$. The constraints on the number of facility types per potential facility location (9), the total number of facilities to be located of each type (10), and the integrality of the decision variables (11) remain the same as those described for the MaxMinMin formulation.

A third formulation can be based on the MaxMinSum objective and seeks to maximize the minimum hub distance for any facility of any type:

$$\text{Max } Z \tag{12}$$

subject to:

$$Z \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{L=1}^{t} Q^{KL} d_{ij} x_j^L + M(1 - x_i^K), \quad i = 1, 2, \ldots, n, \quad K = 1, 2, \ldots, t \tag{13}$$

$$\sum_{K=1}^{t} x_i^K \leq 1, \quad i = 1, 2, \ldots, n \tag{14}$$

$$\sum_{i=1}^{n} x_i^K = p_K, \quad K = 1, 2, \ldots, t \tag{15}$$

$$x_i^K = 0 \text{ or } 1, \quad i = 1, 2, \ldots, n, \quad K = 1, 2, \ldots, t \tag{16}$$

The hub distance associated with each located facility is the sum of the repulsion-weighted distances between that facility and all other located facilities. In this formulation, constraints (13) serve to define the hub distance for each potential facility location i. If no facility of type K is located at i, then Z will be allowed to be a very large sum. When there is a facility of type K located at i, Z is constrained to be less than or equal to the sum of the repulsion-weighted distances between i and all those sites j, where a facility is also located. In those cases where a facility of type L is not located at potential facility location j, then no weighted distance is included in the sum. As p_K facilities of each type must be located, the upper limit on Z will be the minimum repulsion-weighted hub distance (over all facility types and their specific locations). The objective function seeks to maximize this minimum hub distance. Once again the constraints on facility types, number of facilities to locate, and integrality are identical to those in the previous formulations.

The final formulation for the p–t–n dispersion model seeks to maximize the sum of the hub distances from all located facilities to all other located facilities (MaxSumSum).

$$\text{Max} \sum_{i=1}^{n} \sum_{K=1}^{t} \sum_{L=1}^{t} Z_i^{KL} \tag{17}$$

subject to:

$$Z_i^{KL} \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} Q^{KL} d_{ij} x_j^L, \quad i = 1, 2, \ldots, n, \quad K \text{ and } L = 1, 2, \ldots, t \tag{18}$$

$$Z_i^{KL} \leq M x_i^K, \quad i = 1, 2, \ldots, n, \quad K \text{ and } L = 1, 2, \ldots, t \tag{19}$$

$$\sum_{K=1}^{t} x_i^K \leq 1, \quad i = 1, 2, \ldots, n \tag{20}$$

$$\sum_{i=1}^{n} x_i^K = p_K \tag{21}$$

$$x_i^K = 0 \text{ or } 1, \quad i = 1, 2, \ldots, n, \quad K = 1, 2, \ldots, t \tag{22}$$

Rather than determining a single minimum hub distance value as in the MaxMinSum formulation, the MaxSumSum formulation requires that a Z_i^{KL} value be determined for each possible combination of facility types at a given potential facility location i. These minimum hub distances are determined by constraints (18). When a facility of type L is located at facility site j, the minimum repulsion-weighted distance from i to j is included in the sum, weighted by Q^{KL}. Constraints (19) ensure that Z_j^{KL} will only be greater than 0 when a facility of type K is located at facility site i. Consider the example where there are only two types of facilities to locate. In this case, the hub distance for a given facility consists of two Z_I^{KL} values—Z_i^{K1} and Z_i^{K2}. The objective function seeks to maximize the sum of these repulsion-weighted hub distances.

The general models given above identify the basic cases for a family of multiple-type dispersion problems. Within that framework there are a wide range of more specific problems that can be formulated. Due to the limited space available here, only a few of the possible models within the family are presented.

Dispersion layout problems

The layout problem is an extensively studied combinatorial optimization problem that is generally formulated to determine the ideal physical organization for a production system (Meller and Gau 1998). The layout problem is frequently formulated as a quadratic assignment problem (QAP) where every department is assigned to one location and only one department is assigned to any one location. To explain the rationale for a dispersive objective within a layout context, consider the example of different types of chemicals used for processes within a single factory that may have violent reactions when they come in contact with each other. The chemicals must all have a place in the production line, but reactions might best be avoided by maximally dispersing the chemicals inside the factory walls.

Although several dispersion versions of QAP layout formulations have been constructed (Curtin 2002), we present here only a single layout dispersion problem with the MaxMinMin objective where the number of facilities to locate, the number of possible types of facilities, and the number of potential facility locations are all equal. We term this problem the *p–p–p* dispersion problem or the *p*-type dispersion layout problem.

$$\text{Max } Z \tag{23}$$

subject to:

$$Z \leq Q^{KL} d_{ij} + M(2 - x_i^K - x_j^L) \quad i, j, K \text{ and } L = 1, 2, \ldots, p, \quad i \neq j, \quad L \geq K \tag{24}$$

$$\sum_{i=1}^{p} x_i^K = 1, \quad K = 1, 2, \ldots, p \tag{25}$$

$$\sum_{K=1}^{p} x_i^K = 1, \quad i = 1, 2, \ldots, p \tag{26}$$

$$x_i^K = 0 \text{ or } 1, \quad i = 1, 2, \ldots, p, \quad K = 1, 2, \ldots, p \tag{27}$$

This formulation differs from the *p–t–n* dispersion problem in that both the constraints on the number of types of facilities to be located (25) and the number of facilities to locate at each potential facility site (26) are equality constraints. As there are exactly *p* potential facility sites, each site will have a unique facility type located.

Neighborhood constraint elimination

The constraints in the formulations for a family of multiple-type dispersion models consider virtually all of the possible combinations of facility sites and potential types of facilities at those locations. Certain combinations of facility type and sites locations may result in duplicate constraint conditions. It is a relatively straightforward task to eliminate such duplicate conditions. More importantly, it can be shown that for those objectives that seek to maximize minimum distances (MaxMinMin and MaxSumMin) there are certain facility interactions that cannot logically constrain the problem. The removal of these constraints effectively decreases the size of the problem to be solved and therefore would intuitively decrease the time or resources needed to find an optimal solution.

In order to understand this logical elimination of constraints, first consider the case where three facilities of a single type are to be located according to the

MaxMinMin objective. Assume further that the potential facility sites in the region lie along a straight line (see Fig. 3).

The formulation for the general multiple-type dispersion problem with the MaxMinMin objective given above would generate a constraint of type (2) for each (*i*, *j*) pair where *i* is not equal to *j*. As in this simplified instance we are considering a single type of facility, *L* and *K* both will have a value of 1, so additional constraints would not be generated based on differences in type. Within this simplified problem framework consider only the case where a single facility is located at potential facility site, *i*. As three facilities are to be located, there must be two facilities located in addition to site *i*. In order to locate these additional two facilities in such a way as to achieve the greatest possible minimum distance from *i*, they must be located at the potential facility sites labeled b and c in Fig. 3. With this arrangement the maximum minimum distance from *i* to any other located facility is the distance from *i* to b. There is no possible way that the distance from *i* to c will ever be the minimum distance from *i* to another located facility so there is no need to generate a constraint for the pair (*i*, c).

If a total of four facilities are to be located (one at potential facility site *i* and three others) then no constraints are needed for either the pair (*i*, b) or the pair (*i*, c) as neither of those pairings can logically constrain the problem. In general, for a problem with a single type of facility to be located, where *p* facilities are to be located, the (*p*-2) furthest potential facility sites *j* from each potential facility site *i* do not need to be considered when generating constraints. That is, any potential facility site that lies outside the logical neighborhood of site *i* need not be considered. Conversely, constraints need only be generated for the $(n-p+1)$ closest potential facility sites *j* to each potential facility site *i*. In other words, only consider those facilities within the logical neighborhood of *i*. This concept was originally developed in order to streamline the solution of large *p*-median problems (Rosing, Revelle, and Rosing-Vogelaar 1979).

Expanding this concept to problems seeking to optimally disperse multiple types of facilities, the potential facility sites for which constraints need not be

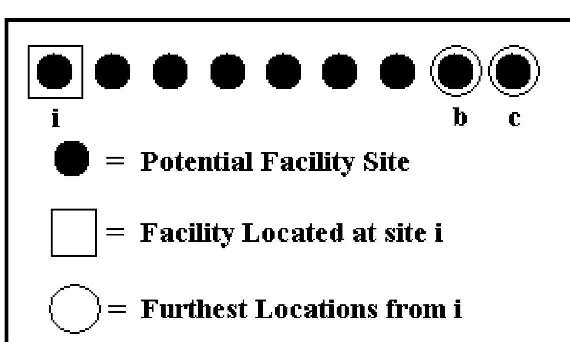

Figure 3. Linear problem demonstrating neighborhood constraint elimination.

generated are not simply those sites that are the furthest distance from a given site i. The repulsion-weighted distance between each i and each j must be considered, and this weighted distance varies with the type of facility being considered for each potential facility site. Therefore, for a facility of type K located at potential facility site i, there are certain combinations of sites j and types L that will have the p-2 greatest weighted distances where j is unique for each combination. Finding these site–type combinations determines the maximum possible minimum distance from i or the boundary of the logical neighborhood of i. However, given that there are multiple types of facilities to be located there may be additional site–type combinations that result in a weighted distance greater than the p-1st furthest site–type combination when j is not unique. One need not generate constraints for any of these site–type combinations.

In order to clarify this concept, consider a problem instance where four facilities of three different types are to be located. Pick a single potential facility site i and assign it a facility of type K. Pick any single one of the remaining sites and call it j. As three types of facilities are being located there are three different possible j^L combinations that can be used to create a repulsion-weighted distance in combination with the site–type i^K and for which constraints would be generated in the absence of constraint elimination. As four facilities are to be located, constraints for the two (p-2) $i^K j^L$ tuples (each tuple being a group of i, j, K, and L in combination) that have the greatest weighted distances can be eliminated. They cannot possibly constrain the objective function value Z. Moreover, these values have defined the logical boundary of the neighborhood of the site–type combination i^K.

To see why this is true, pick another of the remaining potential facility sites j. If any j^L combination has a repulsion-weighted distance from i^K that exceeds the logical boundary set by the previous j under consideration, then those constraints also cannot possibly constrain the solution and can thus be eliminated from the problem formulation. It may in fact be that *all* of the j^L combinations for this second j fall beyond the logical boundary set by the first j. Then consider that this could be true for any of the remaining values of j.

In general, there exists a single logical boundary value (B_i^K) for each i^K combination that can be determined by finding the p-2 smallest maximum distances from that i^K to any j. Any j^L site–type combination that has a weighted distance value greater than those smallest maximum distances lies outside the logical boundary and the associated constraints are unnecessary. While the p-2 rule is used to define this boundary the actual number of constraints that can be eliminated is not a direct function of p, but is instead a function of the distances between potential facility sites and the repulsion measures between facility types.

In order to incorporate the concept of neighborhood constraint elimination into the notation given for the family of multiple-type dispersion problems, consider the set E_i^K which contains those site–type combinations j^L that exceed the maximum possible minimum distance (are outside the logical neighborhood of site *i*). Or:

$$E_i^K = \{j^L | j^L \text{ is outside the logical neighborhood of i}\}$$

The set of constraints (2) in the MaxMinMin formulation and (7) in the MaxSumMin formulation can then be modified to

$$Z \leq Q^{KL} d_{ij} + M(2 - x_i^K - x_j^L), \quad i \text{ and } j = 1, 2, \ldots, n, \quad i \neq j, \; K \text{ and } L = 1, 2, \ldots, t, \; L \geq K, \; j^L \notin E_i^K \tag{28}$$

and

$$Z_i \leq Q^{KL} d_{ij} + M(1 - x_j^L), \quad i \text{ and } j = 1, 2, \ldots, n, \quad i \neq j, \quad K \text{ and } L = 1, 2, \ldots, t, \; j^L \notin E_i^K, \tag{29}$$

respectively. In the following section a comparison is given for both the general formulation and the formulation with neighborhood constraint elimination to demonstrate the differences in problem size when employing this process.

Solving multiple-type dispersion models

Large instances of integer programs such as those formulated in the previous sections can be difficult or impossible to solve optimally. However, using integer programming and branch and bound techniques, optimal solutions can be obtained for small problem instances. Heuristic techniques can be developed to solve larger problem instances. In this section we present the results of general-purpose integer-linear programming software in solving multiple-type facility location dispersion models. We utilized the same data set that was used in Kuby (1987). It consists of 25 points on a Cartesian plane and is depicted in Fig. 4. Interpoint distances are Euclidean.

Using this data set, optimal solutions were sought for a set of 212 multiple-type dispersion problem instances. This set includes 53 instances for each of the four objectives described above. Each set of 53 problems includes instances with values for p ranging from 2 to 10, and values of t ranging from 1 to 9. The lower limit on p is necessary as every solution to any dispersion model that maximizes distance between facilities is trivial when $P = 1$. The lower limit on t (the number of types of facilities to locate) was chosen in order to validate the solution procedure against previously published results, and the upper limit on t was a function of the value of p (in these solutions at least one of each type of facility must be located in order to encourage solutions with multiple types of facilities, so t must be less than p).

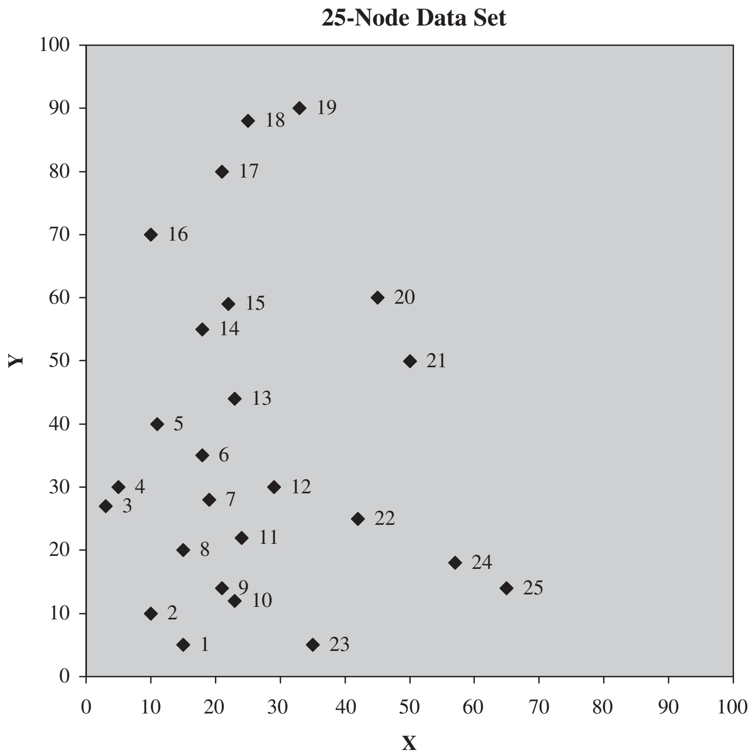

Figure 4. Twenty-five-node data set.

As repulsion measures for facility type pairings must be assigned exogenously with respect to the models, and as the concept of repulsion measures has never previously been implemented, a variety of values were chosen arbitrarily (Table 1).

The problem instances outlined above were formulated using the mathematical programming system format by a program coded in Visual C++. Each problem instance was then solved on a Sun workstation running SunOS 5.7, using CPLEX (version 7.0). In some cases an optimal solution could not be found. This happened either when the memory required for the branch and bound tree exceeded 500 MB or the routine reached a time limit of 1 day.

While it is not reasonable to present the results of solving all 212 problems individually due to space constraints, we will give a summary of the results here. In general, optimal solutions could not be determined for the largest problems attempted. This was true for all of the four objective functions. Unless there are dramatic improvements in either the structure of the problems or the efficacy of the solution procedures, only small instances of multiple-type dispersion problems will be able to be solved optimally using general-purpose linear-integer programming procedures.

As expected, within the problems solved for any of the particular objectives, the solution times generally increase with an increase in the value of either t or p.

Table 1 Repulsion Measures

T	Q^{11}	Q^{12}	Q^{13}	Q^{14}	Q^{15}	Q^{16}	Q^{17}	Q^{18}	Q^{19}	Q^{22}	Q^{23}
2	0.2	0.4								0.6	
3	0.2	0.3	0.4							0.5	0.6
4	0.5	1.0	0.5	0.33						0.5	1.0
5	0.5	1.0	0.5	0.33	0.25					0.5	1.0
6	0.5	1.0	0.5	0.33	0.25	0.2				0.5	1.0
7	0.5	1.0	0.5	0.33	0.25	0.2	0.167			0.5	1.0
8	0.5	1.0	0.5	0.33	0.25	0.2	0.167	0.143		0.5	1.0
9	1000	1.0	0.5	0.33	0.25	0.2	0.167	0.143	0.125	1000	1.0

T	Q^{24}	Q^{25}	Q^{26}	Q^{27}	Q^{28}	Q^{29}	Q^{33}	Q^{34}	Q^{35}	Q^{36}	Q^{37}
2											
3							0.7				
4	0.5						0.5	1.0			
5	0.5	0.33					0.5	1.0	0.5		
6	0.5	0.33	0.25				0.5	1.0	0.5	0.33	
7	0.5	0.33	0.25	0.2			0.5	1.0	0.5	0.33	0.25
8	0.5	0.33	0.25	0.2	0.167		0.5	1.0	0.5	0.33	0.25
9	0.5	0.33	0.25	0.2	0.167	0.143	1000	1.0	0.5	0.33	0.25

T	Q^{38}	Q^{39}	Q^{44}	Q^{45}	Q^{46}	Q^{47}	Q^{48}	Q^{49}	Q^{55}	Q^{56}	Q^{57}
2											
3											
4			0.5								
5			0.5	1.0					0.5		
6			0.5	1.0	0.5				0.5	1.0	
7			0.5	1.0	0.5	0.33			0.5	1.0	0.5
8	0.2		0.5	1.0	0.5	0.33	0.25		0.5	1.0	0.5
9	0.2	0.167	1000	1.0	0.5	0.33	0.25	0.2	1000	1.0	0.5

T	Q^{58}	Q^{59}	Q^{66}	Q^{67}	Q^{68}	Q^{69}	Q^{77}	Q^{78}	Q^{79}	Q^{88}	Q^{89}	Q^{99}
2												
3												
4												
5												
6			0.5									
7			0.5	1.0			0.5					
8	0.33		0.5	1.0	0.5		0.5	1.0		0.5		
9	0.33	0.25	1000	1.0	0.5	0.33	1000	1.0	0.5	1000	1.0	1000

There are several exceptions, however. In the MaxMinSum problem subset those problems where the value of *p* equals the value of *t* are easier to solve than instances with smaller values of *p* and *t*. This may be the case with the MaxSumMin and MaxSumSum objectives as well, but these problems are so difficult to solve optimally that only one example exists in each case to suggest this property. Additionally, for the MaxMinSum objective, those problems with a *t* value of 4 are substantially easier to solve than problems of smaller size. This may be simply a function of the underlying data set, but this property deserves further investigation.

A comparison among the four objectives shows that the MaxMinSum objective—which has the most compact formulation with the smallest number of constraints—is the easiest to solve and consequently has the lowest solution times (Fig. 5). The results suggest that applications of dispersion problems with up to five types of facilities can be solved on small datasets using the formulation presented in this article for the MaxMinSum objective.

Conversely, the MaxSumMin objective function is the most difficult to solve with the largest number of constraints. Even the *p*-dispersion problem where $P = 10$ where only a single type of facility is being located cannot be solved even after processing for a full day. It is therefore not practical to solve multiple-type dispersion problems of even modest size with the MaxSumMin formulation (Fig. 6), using general-purpose optimization software.

The results for the MaxSumMin objective demonstrate the need to test alternate formulations such as the logical neighborhood constraint elimination described in the previous section. When constraint elimination was applied to those problems that had been solved optimally for the MaxMinMin and MaxSumMin objectives, the results were mixed. Figs. 7 and 8 show the percent reduction (or % increase) in solution time with and without neighborhood constraint elimination. Consider first

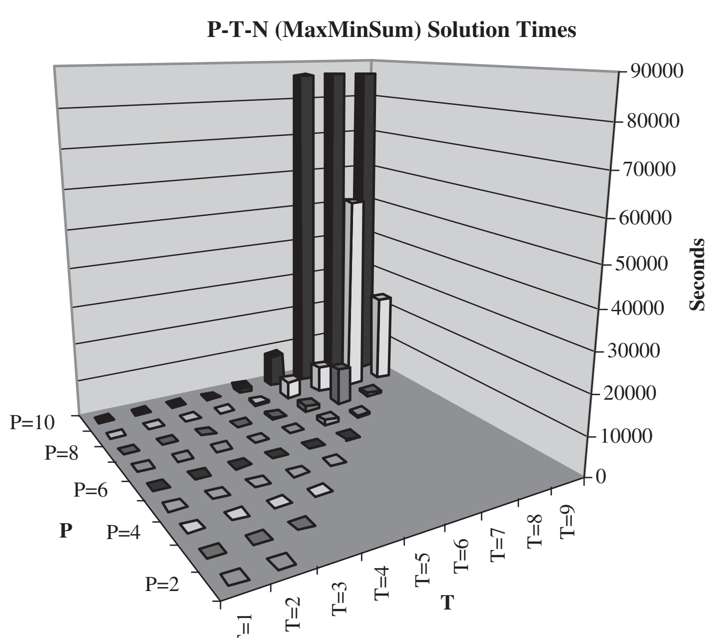

Figure 5. MaxMinSum solution times.

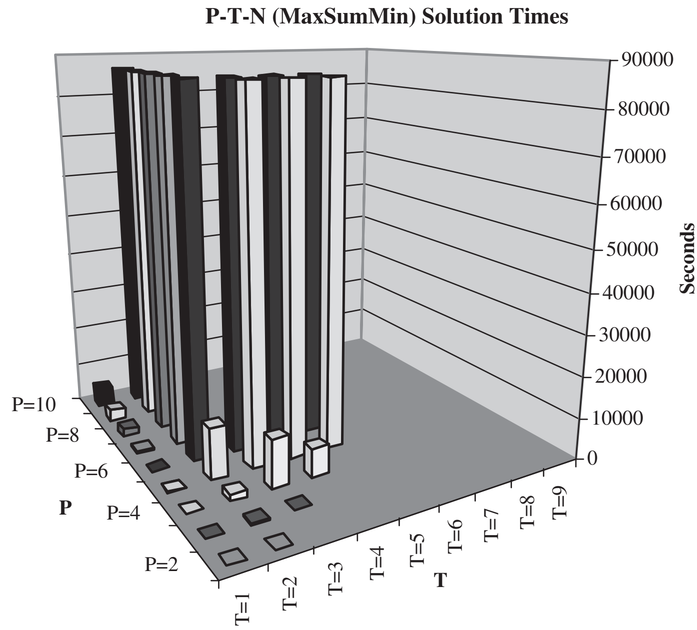

Figure 6. MaxSumMin solution times.

the MaxMinMin objective: of the 41 problems for which optimal solutions have been found, 24 can be solved more quickly when neighborhood constraint elimination has been employed. The improvement ranges from less than 5% to more than 75%. However, 16 problems took longer to solve even though the same

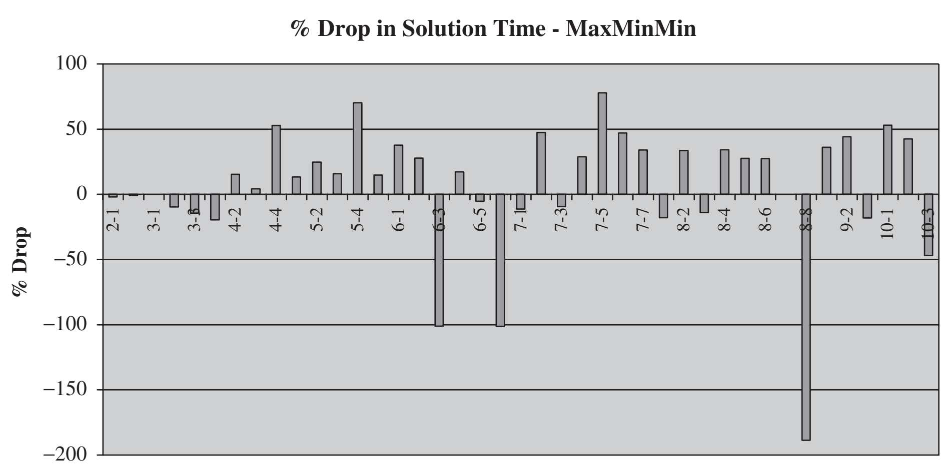

Figure 7. Percent decrease in solution time with neighborhood constraint elimination for the MaxMinMin objective.

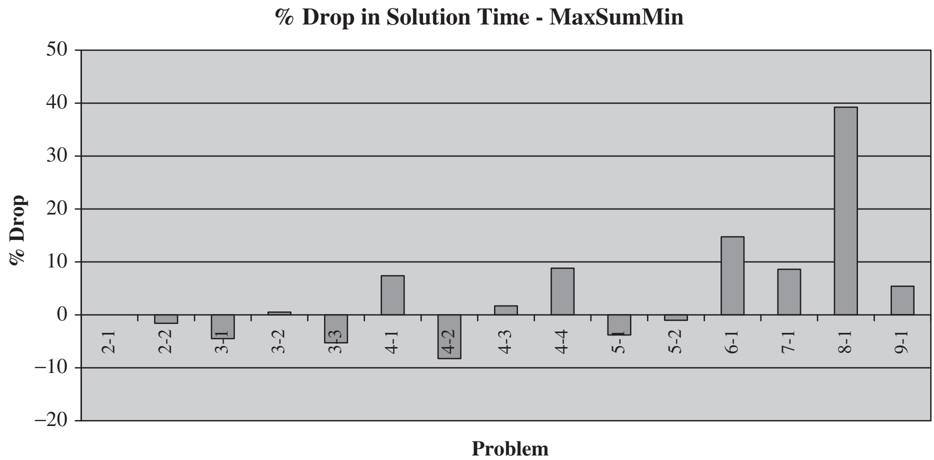

Figure 8. Percent decrease in solution time with neighborhood constraint elimination for the MaxSumMin objective.

number or fewer constraints were used to define the problem. Note that for display purposes the outlying value for the problem where $P = 8$ and $t = 7$ has been eliminated from Fig. 7. This problem took more than eight times longer to solve than it had when no constraint elimination had been applied. Similar results are found for the MaxSumMin objective where 8 of 15 problems were solved more quickly after the application of neighborhood constraint elimination.

As problem size has been reduced using constraint elimination rules for all but the smallest problems with the MaxMinMin and MaxSumMin objectives, it could be expected that all problems would experience a decrease in solution time. It appears that although the reduction in the number of constraints reduces the matrix inversion time, this is sometimes offset by an increase in the branch-and-bound iterations needed to determine integer solutions. While neighborhood constraint elimination is a promising avenue for developing more efficient formulations, the determination of problem instances where it is likely to succeed is an area that requires further research.

Conclusions and future research

This research has addressed a need, identified in the review of the literature surrounding discrete dispersion in location science, for models of discrete dispersion that could deal with multiple types of facilities where maximal dispersion is a function of the interactions among those facilities. This was accomplished through the development of a family of multiple-type dispersion models. Formulations have been provided here that employ four different dispersion objective functions, and several special cases have also been discussed. It has been shown that the spatial structure of the problems can be used to identify a logical neighborhood around potential facilities and to eliminate unnecessary constraints from individual instances of the problem. The range of problem instances that can be solved optimally with a reasonable amount of computational effort was found to be relatively small.

We believe that the family of multiple-type dispersion problems provides a more flexible and robust set of tools for solving location problems where distance should be maximized. This family encompasses the single-type dispersion models that have previously appeared in the literature and treats them as special cases. This expansion should allow for increased diversity in the application of dispersion models. This research has suggested several avenues for future research, including experimentation with different constraint sets (including capacity constraints), the development of heuristics for those problems that cannot be solved optimally using current commercial codes, and the development of a greater understanding involving the nature of interaction and repulsion between different types of facilities.

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